

# Determination of $\alpha_s(1 \text{ GeV})$ from the charmonium fine structure.

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(February 1, 2008)

The strong coupling constant  $\alpha_s(\mu)$  is extracted from the fits to charmonium spectrum and fine structure splittings. The relativistic kinematics is taken into account and relativistic corrections are shown to increase the matrix elements defining spin effects up to 40%. The value of  $\alpha_s(\mu)$  at low-energy scale  $\mu = 1.0 \pm 0.2 \text{ GeV}$  was found to be  $\alpha_s(\mu) = 0.38 \pm 0.03(\text{exp.}) + 0.04(\text{theor.})$  which is about 50% lower than standard perturbative two-loop approximation and is in good agreement with the freezing  $\alpha_s$  behaviour.

11.10.Jj, 11.15.Bt, 12.38.Lg

## I. INTRODUCTION

The strong coupling constant  $\alpha_s(Q^2)$  was measured in many experiments at large momenta where perturbative QCD can be successfully applied [1]. Due to these investigations our knowledge of the fundamental constant  $\Lambda$  in QCD at high energy scale is now much more precise than before. At the low energy scale any extraction of  $\alpha_s$  from experiment is complicated by essential nonperturbative (NP) effects and the  $\tau$  decay is assumed to be the only lowest-energy process from which the coupling constant  $\alpha_s$  can be cleanly extracted: for  $\tau$  decay NP are argued to effects give a small contribution to the hadronic ratio  $R_\tau$  [2]. The averaged value of  $\alpha_s$  at the scale of  $\tau$  lepton mass is now  $\alpha_s(M_\tau) = 0.35 \pm 0.03$  [1].

In the present paper we suggest another piece of low-energy data – very precise experimental measurements of spin splittings of  $\chi_c$  mesons. For reasons discussed below these data can give a unique information on the strong coupling constant  $\alpha_s(\mu)$  at low energy scale,  $\mu \lesssim 1 \text{ GeV}$ . Even though NP effects are essential for  $1P$  charmonium state with the size as large as  $R \approx 0.65 \text{ fm}$  the NP contributions will be shown to affect  $\alpha_s(\mu)$  in a simple and controlled way and the choice of NP parameters is strongly restricted by a fit to charmonium spectrum.

We shall not consider here the hyperfine shift of  $\chi_c$  meson with regard to the center of gravity of  $^3P_j$  multiplet. This shift is small and negative,  $|\Delta_{HF}| < 1 \text{ MeV}$ , and small perturbative (P) and NP contributions cancel in  $\Delta_{HF}$  [3,4]. To explain this shift a very precise theory of NP interaction at small distances is needed which is absent by now.

The experimental magnitude of tensor and spin-orbit splittings on the contrary are large enough ( $> 35 \text{ MeV}$ ) and known with the accuracy better than 2%.

The spin structure of  $P$  states in heavy quarkonia was already investigated several times in the framework of potential approach [4–7]. We quote here the values of  $\alpha_s(\mu)$  obtained in [5,6] from the fit to charmonium fine structure:  $\alpha_s(1.22 \text{ GeV}) = 0.386(\mu \neq m, m = 1.30 \text{ GeV})$  in [5] and  $\alpha_s(\mu = m = 1.20 \text{ GeV}) = 0.35$  in [6]. Note

that however different static interactions were used in [5,6], nevertheless the resulting  $\alpha_s(\mu)$  values at the scale  $\mu \approx 1.2 \text{ GeV}$  are rather close to each other and small in both cases. The calculated value of  $\alpha_s(1.2 \text{ GeV})$  is close to that of  $\alpha_s(M_\tau)$  at the energy scale that is significantly smaller than  $M_\tau$ .

In this paper an improved analysis of the fine structure data will be presented. First, we take into account relativistic kinematics. The wave functions and all matrix elements will be calculated with the help of spinless Salpeter equation as compared to the nonrelativistic unperturbed Hamiltonian used in [5,6]. For the charmonium  $1P$  state  $v^2/c^2 \approx 0.4$  is not small and relativistic corrections are important. For example, whereas the size of  $1P$  state decreases only by  $\sim 8\%$  in relativistic case, the matrix elements like  $\langle r^{-3} \rangle$ ,  $\langle r^{-3} \ln mr \rangle$ , defining spin splittings, increase as much as 40% for any set of parameters. As a result relativistic corrections affect the value of  $\alpha_s(\mu)$  extracted from experimental data.

The second improvement is based on a simple observation. We suggest instead of spin-orbit matrix element  $a = \langle V_{SO}(r) \rangle$  and tensor matrix element  $c = \langle V_T(r) \rangle$  to use their linear combination  $\eta = 3/2 c - a$ . This combination has two remarkable properties:

i) its perturbative part  $\eta_P$  contains only  $\alpha_s^2$  radiative corrections since  $O(\alpha_s)$  terms cancel due to the relation  $c_P^{(1)} = 2/3 a_P^{(1)}$ ;

ii) the parameter  $\eta_P$  does not explicitly depend on the renormalization scale mass  $\mu$ , whereas each of the second order terms,  $c_P^{(2)}$  and  $a_P^{(2)}$ , contains a term proportional to  $\ln \mu/m$  (see Sect. 2).

Due to these features using  $\eta$  instead of spin-orbit matrix element  $a$  is much more convenient while fitting to fine structure data. The experimental values of matrix elements  $a, c$  and  $\eta$  can be easily calculated from  $\chi_c$  meson masses [1]:

$$\begin{aligned} a_{exp} &= 34.56 \pm 0.19 \text{ MeV}, \\ c_{exp} &= 39.12 \pm 0.62 \text{ MeV}, \\ \eta_{exp} &= 24.12 \pm 1.12 \text{ MeV} \end{aligned} \quad (1)$$

Note here that  $\eta_{exp}$  is relatively large ( $\approx 24 \text{ MeV}$ ) with a small experimental error  $< 5\%$ . The cited numbers, Eq. 1, slightly differ from the ones used in [6] because of recent change in  $\chi_{c0}$  mass [1].

Our fitting procedure includes also a fit to charmonium spectrum. Here we prefer more refined fitting to mass level differences than to the absolute values  $M(nL)$  for the given state. The most important mass differences of levels lying below the open charm threshold are

$$\begin{aligned} M(2S) - M(1S) &= 595.39 \pm 1.91 \text{ MeV}, \\ M(1P) - M(1S) &= 457.92 \pm 1.0 \text{ MeV}, \\ M(2S) - M(1P) &= 137.47 \pm 1.77 \text{ MeV} \end{aligned} \quad (2)$$

Here  $M(nL)$  is a spin-averaged mass.

## II. PERTURBATIVE FINE STRUCTURE PARAMETERS

Every matrix element  $a, c$  or  $\eta$  includes both P and NP contributions:  $c = c_P + c_{NP}$ , etc. First, P interaction will be discussed. The spin-dependent P interaction is now known in one-loop approximation only. Therefore our analysis can be only done with  $O(\alpha_s^2)$  terms. In coordinate space the spin-dependent interaction including  $\alpha_s^2$  corrections in the renormalization  $\overline{MS}$  scheme was calculated in [8]. Matrix elements of spin-orbit and tensor interactions for a number of flavors  $n_f = 3$ , can be easily found from the potentials [8],

$$a_P = a_P^{(1)} + a_P^{(2)}, \quad a_P^{(1)} = \frac{2\alpha_s(\mu)}{m^2} \langle r^{-3} \rangle, \quad (3)$$

$$\begin{aligned} a_P^{(2)} &= \frac{2\alpha_s^2(\mu)}{\pi m^2} \left\{ 4.5 \langle r^{-3} \rangle \ln \frac{\mu}{m} + \right. \\ &\quad \left. 1.58193 \langle r^{-3} \rangle + 2.5 \langle r^{-3} \ln(mr) \rangle \right\} \end{aligned} \quad (4)$$

$$c_P = c_P^{(1)} + c_P^{(2)}, \quad c_P^{(1)} = \frac{4}{3} \frac{\alpha_s(\mu)}{m^2} \langle r^{-3} \rangle. \quad (5)$$

$$\begin{aligned} c_P^{(2)} &= \frac{4}{3} \frac{\alpha_s^2(\mu)}{\pi m^2} \left\{ 4.5 \langle r^{-3} \rangle \ln \frac{\mu}{m} + \right. \\ &\quad \left. 3.44916 \langle r^{-3} \rangle + 1.5 \langle r^{-3} \ln(mr) \rangle \right\} \end{aligned} \quad (6)$$

In Eqs. (3–6) all matrix elements will be calculated for  $1P$  state. From Eq. (4) and Eq. (6) one can see that  $a_P^{(2)}$  and  $c_P^{(2)}$  contain  $\ln \mu/m$  with the large coefficient whenever  $\mu \neq m$ . However in the linear combination  $\eta_P = 3/2 c_P - a_P$  these terms are cancelled and the following simple expression is obtained for  $\eta_P$ ,

$$\eta_P = \frac{3}{2} c_P^{(2)} - a_P^{(2)} = \frac{2f_0}{\pi m} \alpha_s^2(\mu), \quad (7)$$

where  $f_0$  is the combination of matrix elements,

$$f_0 = (m)^{-1} \{ 1.86723 \langle r^{-3} \rangle - \langle r^{-3} \ln mr \rangle \}, \quad (8)$$

which has a weak dependence on parameters of static interaction and charm-quark mass. Practically in all cases  $f_0 = 0.12 \pm 0.01$ .

The simple connection between  $\alpha_s^2(\mu)$  and  $\eta_P$ , Eq. (7), will be used later in our fit to the experimental fine structure parameter  $\eta_{exp}$ , given by Eq. (1).

## III. STATIC INTERACTION

All spin effects in charmonium are very small as compared to the level masses or mass level differences in Eq. (2) and therefore can be considered as a perturbation. At this point the choice of unperturbed Hamiltonian is of importance. In [6] Schrödinger Eq. with the Cornell potential was used for this purpose. Here instead we take relativistic spinless Salpeter equation (SSE),

$$\left[ 2\sqrt{\vec{p}^2 + m^2} + V_0(r) \right] \psi_{nl}(r) = M_{nl} \psi_{nl}(r) \quad (9)$$

In the framework of potential model this equation with square root kinetic term was successfully used in many calculations of meson masses and properties during last twenty years [9,10]. Recently it was deduced directly from QCD under a assumption of area law of Wilson loop in the framework of proper-time Feynman-Schwinger approach [11]. Therefore SSE, Eq. (9), cannot be consider as an *ad hoc* potential model, but rather on the same grounds as the QCD sum rules approach.

In Eq. (9) the charm-quark mass  $m$  is a pole mass defined by the pole of quark propagator [7]. The pole mass  $m$  is related to the running mass in  $\overline{MS}$  scheme,  $m(\overline{m}^2)$ , as [7]

$$\begin{aligned} m \equiv m_{pole} &= \overline{m}(\overline{m}^2) \left\{ 1 + \frac{4}{3} \frac{\alpha_s(m^2)}{\pi} + \right. \\ &\quad \left. \left( K - \frac{8}{3} \right) \left( \frac{\alpha_s(m^2)}{\pi} \right)^2 \right\} \end{aligned} \quad (10)$$

For  $c$  quark  $K \approx 14.0$ . From Eq. (10) one can estimate that for example for  $m = 1.4 \text{ GeV}$  and  $\alpha_s(m) \approx 0.3$  the relation  $m/\overline{m} \approx 1.23$ , i.e.  $m$  is about  $20 \div 30 \%$  larger than  $\overline{MS}$  mass  $\overline{m}$ . The static potential  $V_0(r)$  in Eq. (9) is taken here as Coulomb potential plus linear confining term, as it was done in [6],

$$V_0(r) = -\frac{4}{3} \frac{\tilde{\alpha}_V(\mu)}{r} + \sigma r \quad (11)$$

Here the vector coupling constant  $\tilde{\alpha}_V(\mu)$  differs from the running constant  $\alpha_s(\mu)$  in  $\overline{MS}$  scheme. The connection between them was found in [12] and for  $n_f = 3$  it reads

$$\tilde{\alpha}_V(\mu) = \alpha_s(\mu) \left[ 1 + \frac{1.75}{\pi} \alpha_s(\mu) \right] \quad (12)$$

In Eq. (11)  $\tilde{\alpha}_V(\mu)$  will be taken at some fixed point  $\mu$  which in general can differ from the scale  $\mu_0$  which defines spin splittings because Coulomb interaction behaves as  $r^{-1}$  whereas spin-dependent interaction behaves as  $r^{-3}$  and therefore is more sensitive to smaller distances. However, in our calculations it was found that the choice  $\mu \approx \mu_0$  together with the additional condition, Eq. (12), gives rise to a good description of both charmonium spectrum and fine structure.

In Eq. (11) confining potential was chosen to be linear at all distances, as it is done in most papers. This is in agreement with direct lattice calculations of static interaction [14], but is at variance with OPE and field correlator approach, which require that the NP interaction should start at small  $r$  as  $const \cdot r^2$  and tend to linear form  $\sigma r$  only at  $r > T_g$ , where  $T_g$  is the gluonic correlation length, measured on the lattice in [15] to be  $T_g \approx 0.3 \text{ fm}$ .

Recently some arguments have been given in favour of additional linear potential at small  $r$  [16] and it was explicitly found from P-NP interference in [17] to be of the magnitude which effectively resolves the discrepancy and confirm the linear NP potential at all  $r$ .

Therefore we assume below that linear potential  $\sigma r$  is valid at all distances. To control the dependence of our results on the choice of string tension the parameter  $\sigma$  will be varied in some interval compatible with a good description of mass level differences in charmonium.

For pure linear potential NP contribution to spin-orbit potential is given by Thomas interaction for which

$$a_{NP} = -\frac{\sigma}{2m^2} \langle r^{-1} \rangle, \quad a_{NP} < 0 \quad (13)$$

The value of  $c_{NP}$  is defined by  $\mathcal{D}_1$ -correlator [13,18] which is small in lattice calculations [15], yielding the estimate  $0 < c_{NP} < 0.3 \text{ MeV}$  [18]. This result is in a good agreement with direct lattice calculations of NP tensor potential [19], where  $V_T(r)$  was found at the distances  $0.2 \lesssim r \lesssim 0.6 \text{ fm}$  to be  $V_T(r) < 1 \text{ MeV}$ . Thus value of  $c_{NP}$  is much less than  $|a_{NP}| \sim 15 \div 20 \text{ MeV}$  and therefore can be neglected in  $\eta_{NP}$ . The theoretical error due to neglected term in Eq. 14 is small and will be taken into account in our analysis. Then

$$\eta_{NP} = \frac{3}{2} c_{NP} - a_{NP} \approx \frac{\sigma}{2m^2} \langle r^{-1} \rangle \quad (14)$$

To calculate wave functions of SSE the expansion of  $\psi_{nl}$  in a series over Coulomb-type functions, suggested in [9], was used. The numerical calculations for  $1P$  state provided high accuracy (better than  $10^{-3}$ ) for matrix elements like  $\langle r^{-1} \rangle$ ,  $\langle r^{-3} \rangle$ , etc.. Some of them are listed in Table I both for SSE and nonrelativistic Schrödinger equation for three sets of parameters. Set A with  $m = 1.2 \text{ GeV}$ ,  $\sigma = 0.2 \text{ GeV}^2$  and  $\tilde{\alpha}_V = 0.35$  was taken from the paper [6]. From Table I one can see that the difference between relativistic (R) and nonrelativistic (NR) matrix elements is about 8% for square

root radius,  $\sim 10\%$  for matrix element  $\langle r^{-1} \rangle$ , defining NP effects, Eq. (13), and very large,  $\sim 40\%$ , for matrix element  $\langle r^{-3} \rangle$ . This growth of  $\langle r^{-3} \rangle$  is due to the decreasing  $1P$ -state size in relativistic case.

One should also note here that mass difference  $M(2S) - M(1P)$  is large in NR case ( $\sim 170 \text{ MeV}$ ), i.e.  $\sim 20\%$  larger than the experimental value  $137 \text{ MeV}$ , Eq. (2). When relativistic kinematic is taken into account then excellent agreement with experiment can be easily reached for this mass difference, see Table II.

#### IV. FITTING CONDITIONS

Our fitting procedure includes two conditions: (I)  $\eta = \eta_{exp}$  and (II)  $c = c_{exp}$ . Using Eqs. (7) and (14) we write the first condition as

$$\begin{aligned} \eta &= \frac{3}{2} c_P^{(2)} - a_P^{(2)} + \frac{\sigma}{2m^2} \langle r^{-1} \rangle = \\ &= \eta_{exp} = (24.12 \pm 1.12) 10^{-3} \text{ GeV} \end{aligned} \quad (15)$$

With the help of Eqs. (4) and (6) this condition can be rewritten in another form,

$$\alpha_s^2(\mu) \cdot \frac{2f_0}{\pi m} = (24.12 \pm 1.12) 10^{-3} - \frac{\sigma}{2m^2} \langle r^{-1} \rangle \equiv \Delta \quad (16)$$

From here  $\alpha_s(\mu)$  can be defined through the known numbers,

$$\alpha_s(\mu) = \sqrt{\frac{\pi m \Delta}{2f_0}} \quad (17)$$

where  $m, \sigma$  are fixed parameters and  $f_0$  is given by the expression in Eq. (8). Our calculations show that  $f_0$  is almost constant,  $f_0 \approx 0.12 \pm 0.01$ , independently on the choice of other parameters of static interaction,  $m, \sigma$  and  $\tilde{\alpha}_V$ .

As seen from Eq. (17) the coupling constant is proportional to  $\sqrt{m}$  and its value also strongly depends on  $\sqrt{\Delta}$ , where  $\Delta$  in Eq. (16) is the difference between  $\eta_{exp}$  and the absolute value of NP spin-orbit matrix element  $|a_{NP}|$ . As it follows from our calculations this difference can become negative for some small quark masses ( $m \lesssim 1.3 \text{ GeV}$ ) and large  $\sigma \approx 0.2 \text{ GeV}^2$  whereas the l.h.s. of the Eq. (16) is always positive. Therefore the solutions with such small masses and large  $\sigma$  must be excluded from our fit to the experimental data. For example, the Set A from [6] considered in the relativistic case yields the parameter  $|a_{NP}| = 26.53 \text{ MeV}$  which is larger than  $\eta_{exp}$  in Eq. (1) and hence inappropriate.

With the use of Eq. (17)  $\alpha_s(\mu)$  can be precisely determined for the given set of parameters  $m$  and  $\sigma$ . At this stage the scale  $\mu$  still remains undefined but  $O(\alpha_s)$  terms,  $c_P^{(1)}$  and  $a_P^{(1)}$ , which do not explicitly depend on  $\mu$ , can be calculated.

At the second step one can fit the second condition,  $c = c_{exp} = 39.12 \pm 0.62 \text{ MeV}$ , where  $c = c_P^{(1)} + c_P^{(2)} + c_{NP}$ . As it was discussed above NP term  $c_{NP}$  is small,  $c_{NP} < 0.3 \text{ MeV}$ , and as compared to  $c$  can be neglected. With  $c_P^{(1)}$  already calculated above, Eq. (5), the second condition can be rewritten as

$$c_P^{(2)} = c_{exp} - \frac{4}{3} \frac{\alpha_s(\mu)}{m^2} \langle r^{-3} \rangle \quad (18)$$

Using Eq. (6) it can be represented as

$$\begin{aligned} \frac{4}{3} \frac{\alpha_s(\mu)}{\pi m^2} \left\{ \langle r^{-3} \rangle [3.44916 + 4.5 \ln \frac{\mu}{m}] + \right. \\ \left. 1.5 \langle r^{-3} \ln(mr) \rangle \right\} = \\ (39.12 \pm 0.62) 10^{-3} - \frac{4}{3} \frac{\alpha_s(\mu)}{m^2} \langle r^{-3} \rangle \end{aligned} \quad (19)$$

All matrix elements and  $\alpha_s(\mu)$  in Eq. (19) are already known for given  $m$ ,  $\sigma$  and  $\tilde{\alpha}_V$ , therefore Eq. (19) fixes  $\mu$  at some value  $\mu_0$ . It was found in our numerous calculations that the value  $\mu_0$  does not coincide with the  $c$  quark mass  $m$  in general case, and only for very special choice of charm quark mass and static potential parameters  $\tilde{\alpha}_V$  and  $\sigma$  the condition  $\mu = m$  can be satisfied. That might be the reason why the choice  $\mu = m$  apriori taken in the paper [6] has difficulties with simultaneous fitting of spin-orbit and tensor splittings (for  $1P$  state) with the same  $\alpha_s(\mu)$  if  $c$  quark mass  $m = 1.8 \text{ GeV}$ . In our fit, without the choice  $\mu = m$ , when the Eq. (16) and Eq. (19) are satisfied, we "automatically" get  $a = a_{exp}$ .

Hence for fixed  $m$  and  $\sigma$  one finds  $\alpha_s(\mu)$  and  $\mu = \mu_0$  satisfying experimental data. In the next Section we shall check whether this choice satisfies also data on the whole spectroscopy of charmonium.

## V. THE CHOICE OF PARAMETERS

We shall discuss here only those fits with given  $m$ ,  $\sigma$  and  $\tilde{\alpha}_V$  which yield good description of spin-averaged spectrum in charmonium. As a result there appear some restrictions on the magnitude of  $\sigma$  and  $\tilde{\alpha}_V$ , but the choice of pole mass  $m$  remains relatively arbitrary. Even if one takes  $\overline{MS}$  mass,  $\overline{m} = 1.30 \pm 0.20 \text{ GeV}$ , as it is commonly accepted [1], and makes use of the Eq. (10), then the values of the pole mass  $m$  can vary from  $m_{min} \approx 1.2 \text{ GeV}$  to  $m_{max} \approx 1.8 \text{ GeV}$ . Therefore it is important to impose additional physical restrictions on  $m$ . From our fitting procedure it is clear that for any mass  $m$  the corresponding  $\alpha_s(\mu_0)$  and  $\mu_0$  can be formally found. However the dependence of  $\alpha_s(\mu)$  on  $\mu$  is different for different  $m$ . Some restrictions on the value of pole quark mass come from our fit to the fine structure data.

All considered solutions with different  $m$  can be separated in three groups:

i) Quark pole mass is small,  $m \lesssim 1.3 \text{ GeV}$ . Then in Eq. (16) the difference  $\Delta \approx 2 \div 4 \text{ MeV}$  is also small

since NP spin-orbit matrix element  $|a_{NP}|$ , proportional to  $m^{-2}$ , is large. The value of  $\Delta$  remains small even for the small  $\sigma \approx 0.17 \text{ GeV}$ . Then according to Eq. (17)  $\alpha_s(\mu) \sim \sqrt{m\Delta}$  is small,  $\alpha_s(\mu) \lesssim 0.20$ . As a consequence,  $O(\alpha_s)$  terms,  $c_P^{(1)}$  and  $a_P^{(1)}$ , are not large compared to the second order terms  $c_P^{(2)}$  and  $a_P^{(2)}$  and in some cases  $c_P^{(2)} > c_P^{(1)}$  were obtained. For small  $m$  the calculations give large value of scale,  $\mu \gtrsim 3 \text{ GeV}$ , and in many cases  $\mu \gtrsim 10 \text{ GeV}$ , so that in all cases  $\ln \mu/m \gtrsim 1$ . Note that the pole mass  $m \lesssim 1.3 \text{ GeV}$  corresponds to the  $\overline{MS}$  mass  $\overline{m} \lesssim 1.15 \text{ GeV}$ .

Therefore for small  $m$  and any  $\sigma$  the fine structure parameters  $a$  and  $c$  strongly depend on  $\mu$ , and  $\alpha_s^2$  corrections give large or even dominant contribution. Those solutions will be excluded in our analysis as unphysical.

ii) The pole mass  $m$  is large,  $m \gtrsim 1.6 \text{ GeV}$ , which corresponds  $\overline{MS}$  mass  $\overline{m} \gtrsim 1.3 \text{ GeV}$ . Then in contrast to small  $m$  case the value  $\Delta \approx 10 \div 12 \text{ MeV}$  is large since the matrix element  $|a_{NP}|$  is smaller for large  $m$ . Therefore  $\alpha_s(\mu) \sim \sqrt{m\Delta}$  grows large,  $\alpha_s(\mu) \gtrsim 0.42$  in our calculations. However, for large  $m$  the value of  $\mu$  was found to be small,  $\mu \lesssim 0.7 \text{ GeV}$ , so that  $|\ln \mu/m| \sim 1$  is large again. As a consequence the negative term, proportional to  $\ln \mu/m$  in  $c_P^{(2)}$ , Eq. (6), and  $a_P^{(2)}$ , Eq. (4), cancel positive contribution from other two terms, and as a result  $O(\alpha_s^2)$  terms are numerically rather small. In some cases  $a_P^{(2)}$  even becomes negative.

Thus the second order terms have strong dependence on  $\mu$  and one gets relatively large value of  $\alpha_s(\mu) \approx 0.4 \div 0.5$  at small mass scale  $\mu \lesssim 0.7 \text{ GeV}$ . Therefore in this case one can expect large contribution from  $\alpha_s^3$  corrections which are still unknown and hence any decisive conclusions about those solutions with large  $m$  are now impossible.

iii) The pole quark mass is in the range  $1.4 \div 1.56 \text{ GeV}$ . For those masses  $\mu$ -dependent term in  $c_P^{(2)}$  and  $a_P^{(2)}$  does not dominate and typically  $|\ln \mu/m| \lesssim 0.4$  or  $0.6 \lesssim \mu/m \lesssim 1.0$ . Just for such masses the best fit to charmonium spectrum was obtained.

From two fits, to the spectrum and the fine structure splittings, we have found out the following restrictions on the choice of  $m$  and  $\sigma$ ,

$$m = 1.48 \pm 0.08 \text{ GeV}, \quad \sigma = 0.178 \pm 0.008 \text{ GeV}^2 \quad (20)$$

For such  $m$  and  $\sigma$  the Coulomb constant  $\tilde{\alpha}_V$  is "automatically" fixed by the fit to charmonium spectrum:  $\tilde{\alpha}_V = 0.42 \pm 0.04$ . Note also that the pole mass  $m$  in the range given by Eq. (20) corresponds to the  $\overline{MS}$  mass  $\overline{m} \approx 1.18 \pm 0.07 \text{ GeV}$ .

With the parameters from Eq. (20) the calculated values of  $\mu_0$  are in range,  $\mu_0 \approx 1.0 \pm 0.2 \text{ GeV}$ . At this scale,  $\mu_0 \sim 1 \text{ GeV}$ , the extracted value of  $\alpha_s(\mu_0)$  (with experimental and theoretical errors) is

$$\alpha_s(\mu_0) = 0.38 \pm 0.03(exp) \pm 0.04(theory) \quad (21)$$

$$0.75 \leq \mu_0 \leq 1.2 \text{ GeV} \quad (22)$$

Our calculations for two different sets of parameters are represented in Table II.

Set D with  $m = 1.4 \text{ GeV}$ ,  $\sigma = 0.183 \text{ GeV}^2$ , and  $\tilde{\alpha}_V = 0.39$  was selected because for this variant  $\mu_0 \approx m$  was obtained, i.e. the term with  $\ln \mu/m$  does not contribute to  $c_P^{(2)}$  and  $a_P^{(2)}$ . Also for Set D a good description of charmonium spectrum (see Table II) was obtained. From the fine structure fit the extracted  $\alpha_s(\mu_0)$  and  $\mu_0$  values are

$$\alpha_s(\mu_0) = 0.312, \mu_0 = 1.36 \text{ GeV} \approx m \quad (23)$$

This  $\alpha_s(\mu_0)$  is small as compared with  $\alpha_s(M_\tau)$  extracted from  $\tau$  decay [1,2]. If we put the additional restriction:

$$\alpha_s(m < M_\tau) > \alpha_s(M_\tau) = 0.35 \pm 0.03 \quad (24)$$

then we have to conclude that for Set D the Eq. (24) is violated.

To control the dependence of  $\alpha_s(\mu)$  on NP parameter  $\sigma$  we have calculated  $\alpha_s(\mu)$  for smaller  $\sigma$ , varying  $\sigma$  till the description of charmonium spectrum was becoming poor. For example, for  $\sigma = 0.17 \text{ GeV}^2$   $M(2S) - M(1S)$  and  $M(1P) - M(1S)$  were already 30 MeV lower than its experimental values. For this  $\sigma = 0.17 \text{ GeV}^2$  ( $m = 1.40 \text{ GeV}$  and  $\tilde{\alpha}_V = 0.39$ ) the extracted  $\alpha_s(\mu_0)$  is increasing,

$$\alpha_s(\mu_0) = 0.373, \mu_0 = 0.90 \text{ GeV}, \quad (25)$$

these fitted values lie exactly in the range given by Eq. (21) and Eq. (22).

The best fit in our calculations was found for the set of parameters C,

$$\begin{aligned} \text{Set C: } m &= 1.48 \text{ GeV}, \\ \sigma &= 0.18 \text{ GeV}^2, \tilde{\alpha}_V = 0.42. \end{aligned} \quad (26)$$

Note that the pole mass  $m = 1.48 \text{ GeV}$  corresponds to  $\bar{m} \approx 1.17 \text{ GeV}$ . For Set C the excellent agreement for the mass level differences in charmonium (see Table II) was obtained and

$$\alpha_s(\mu_0) = 0.365 \pm 0.027(\text{exp}), \mu_0 = 0.91 \text{ GeV} \quad (27)$$

To estimate the theoretical error coming from NP effects we have varied  $\sigma$  and analyzed the  $\alpha_s$  dependence on  $\sigma$ . For the smaller  $\sigma$ ,  $\sigma = 0.17 \text{ GeV}^2$ ,  $\alpha_s(\mu)$  is increasing,

$$\alpha_s(\mu_0) = 0.408 \pm 0.027(\text{exp}), \mu_0 = 0.94 \text{ GeV} \quad (28)$$

For this value of  $\sigma$  the fit to spectrum is worse as compared to  $\sigma = 0.18 \text{ GeV}^2$ , in particular mass level differences are about 5 ÷ 10% less than the experimental ones, Eq. (2). Therefore  $\alpha_s(\mu)$  given by Eq. (28) can be considered as the upper limit for the coupling constant  $\alpha_s(\mu)$  with  $m = 1.48 \text{ GeV}$ . Averaging two numbers,

Eq. (27) and Eq. (28), yields the extracted value of  $\alpha_s(\mu)$  in Eq. (21).

One should be reminded here that the nonrelativistic analysis in [5] discussed in Introduction has given  $\alpha_s(1.22) = 0.386$  (and larger  $\mu = 1.22 \text{ GeV}$ ) close to our resulting value, Eq. (21), however, in relativistic case the fitted value of  $\alpha_s(\mu)$  would be smaller for the same parameters  $m$ ,  $\sigma$  and  $\tilde{\alpha}_V$  used in nonrelativistic approach in [5]. Here we would like also to note a remarkable agreement between our result Eq. (21) and the value  $\alpha_s(\mu_0) = 0.38 \pm 0.05$  ( $\mu_0 = 0.93 \text{ GeV}$ ) extracted from the best overall fit to  $2P$  state in botomonium [4,7]. This coincidence is probably not occasional. Both systems,  $2P \text{ } b\bar{b}$  state and  $1P \text{ } c\bar{c}$  state, have exactly the same size:

$$\sqrt{\langle r^2(b\bar{b}) \rangle_{2P}} = \sqrt{\langle r^2(c\bar{c}) \rangle_{1P}} = 0.63 \pm 0.03 \text{ fm}.$$

The extracted value  $\alpha_s(0.9) \approx 0.38$  turns out to be smaller then the corresponding value in PQCD. If we take for  $n_f = 5$ ,  $\Lambda^{(5)} = 237_{-24}^{+26} \text{ MeV}$  in two-loop approximation from recent compilation [1] and calculate  $\Lambda^{(4)}$  and  $\Lambda^{(3)}$  from the matching conditions at  $\overline{MS}$  quark masses,  $\bar{m}_b = 4.3 \text{ GeV}$  and  $\bar{m}_c = 1.3 \text{ GeV}$ , then  $\Lambda^{(4)} = 338_{-31}^{+33} \text{ MeV}$  and  $\Lambda^{(3)} = 384.4_{-30}^{+32} \text{ MeV}$ . Then in two-loop approximation  $\alpha_s(1 \text{ GeV}) = 0.54_{-0.05}^{+0.06}$  is about a factor 1.5 larger than  $\alpha_s(1 \text{ GeV}) \approx 0.36 \pm 0.03 \pm 0.03$  in our fit, the behaviour of perturbative  $\alpha_s(\mu)$  is shown as a band in Fig. 1. Seven points from our fits with different  $c$  quark masses,  $1.40 \leq m \leq 1.48$  are also shown in Fig. 1.

The variation of quark mass gives rise to changes in the values of  $\mu$  and as a result we get the dependence of  $\alpha_s(\mu)$  on the scale  $\mu$  in low-energy region. It is important to underline that for  $1P$  state in charmonium the scale parameter  $\mu$  lies in low-energy region,  $\mu \lesssim 1 \text{ GeV}$ , in all our fits (our seven points are shown in Fig. 1)  $\alpha_s(\mu) \lesssim 0.42$  is in agreement with Eq. (21). In all cases the extracted values of  $\alpha_s(\mu)$  are below those in perturbation theory.

How to interpret this result? We cannot accept the point of view that the relatively small value of  $\alpha_s(1 \text{ GeV})$  found here points out that  $\Lambda^{(3)}$  should be smaller than the cited above perturbative value of  $\Lambda^{(3)} \approx 384 \text{ MeV}$  which corresponds to  $\alpha_s(M_Z) = 0.119 \pm 0.002$ . In our fit to the charmonium fine structure we have met a very specific case when  $\alpha_s(\mu)$  is extracted at low-mass scale,  $\mu \lesssim 1 \text{ GeV}$ . In this region the  $\alpha_s(\mu)$  freezing phenomenon is expected [20] which can affect the resulting values of the strong coupling constant, as is discussed below.

## VI. $\alpha_s$ FREEZING

Perturbative evolution of  $\alpha_s$  at large distances is physically modified due to NP fields, which create confining strings (and hence NP mass parameter) and make an effective large-distance cut-off in all loop integrals. This mechanism leads to a specific regime of  $\alpha_s$  evolution at small  $Q$ , which is called freezing, and was studied both theoretically [20] and in experimental analysis [21].

Theoretically the freezing of the strong coupling constant can be deduced if one takes into account the behaviour of running coupling constant in background vacuum fields [20]. Experimentally phenomenon of freezing was anticipated already for some time (see review in [21]).

The coupling constant, denoted as  $\alpha_B(q^2)$ , appears to be an analytical function at all Euclidean momenta, because it depends on  $q^2 + m_B^2$ , where  $m_B^2$  is a background mass. At large  $q^2 \gg \Lambda_B^2$   $\alpha_B(q^2)$  coincides with perturbative  $\alpha_s(q^2)$ . It means that  $\Lambda_B^{(5)} = \Lambda^{(5)}$  and  $\alpha_B(q^2)$  keeps the property of asymptotic freedom. At small  $q^2$ ,  $\alpha_B(q^2)$  is freezing at some constant value, which depends on the background mass  $m_B$  [22]. The behaviour of  $\alpha_B(q^2)$  in two-loop approximation is given by the following approximate expression [20,22,23]:

$$\alpha_B(q^2) = \frac{4\pi}{\beta_0 t_B} \left( 1 - \frac{\beta_1}{\beta_0^2} \frac{\ln t_B}{t_B} \right) \quad (29)$$

with  $t_B = \ln \frac{q^2 + m_B^2}{\Lambda_B^2}$

The background mass  $m_B$  in Eq. (29) can depend on the process considered and for static interquark interaction  $m_B$  coincides with the lowest hybrid mass [20,22]. Analytic and lattice calculations [24] predict the energy of the lowest hybrid excitation in the interval  $1 \div 1.5 \text{ GeV}$ . In that follows we find the best fitting value  $m_B = 1.1 \text{ GeV}$ , which is well inside the predicted interval.

Since  $\Lambda_B^{(5)} = \Lambda^{(5)}$  and  $m_B$  is fixed,  $\Lambda_B^{(4)}$  and  $\Lambda_B^{(3)}$  can be calculated from the matching conditions at the flavor thresholds ( $\bar{m}_b = 4.3 \text{ GeV}$  and  $\bar{m}_c = 1.3 \text{ GeV}$ ). The obtained values of  $\Lambda_B^{(4)}$  and  $\Lambda_B^{(3)}$  turn out to be very close to perturbative values. In two-loop approximation  $\Lambda_B^{(4)} = 339_{-31}^{+33} \text{ MeV}$  is only by  $1 \text{ MeV}$  larger than  $\Lambda_B^{(4)}$  and  $\Lambda_B^{(3)} = 400_{-31}^{+33} \text{ MeV}$  is by  $16 \text{ MeV}$  larger than  $\Lambda^{(3)}$ .

The freezing value of  $\alpha_B(0) = 0.5_{-0.05}^{+0.06}$  can be found from the definition Eq. (29) with  $m_B = 1.1 \text{ GeV}$ . The behaviour of  $\alpha_B(\mu)$  as a function of  $\mu$  is shown in Figure 1. by the dashed curve. From Figure 1 one can see that all seven values of  $\alpha_s(\mu_0)$  found in our fit with different  $m$  lie on this curve  $\alpha_B(\mu)$  in the region  $\mu \lesssim 1.2 \text{ GeV}$ .

At the scale of the  $\tau$  lepton mass the value of  $\alpha_B(M_\tau) = 0.306_{-0.014}^{+0.015}$  is close to the lower limit of  $\alpha_s(M_\tau) = 0.35 \pm 0.03$  from the compilation [1]. However, in some theoretical approaches, e.g. in renormalon chain model, the value of  $\alpha_s(M_\tau) = 0.305$  is preferred [2], which coincides with our prediction.

Note also that  $\alpha_s$  freezing phenomena is actually observed in recent lattice calculations of three gluon vertex function in [25] where  $\alpha_s(\mu)$  points at  $\mu < 1.8 \text{ GeV}$  lie far below than perturbative  $\alpha_s(\mu)$  values and all points look like going to some constant value.

## VII. CONCLUSION

Two fits – to mass level differences and the fine structure splittings in charmonium were done here. In all calculations the relativistic kinematics was taken into account which is more important for spin-dependent effects than for the spin-averaged  $c\bar{c}$  spectrum.

The fine structure parameters experimentally known from precise measurements of  $\chi_c$  masses are not small ( $\sim 35 \text{ MeV}$ ) and measured with accuracy better 2%. The value of linear combination  $\eta = 3/2c - a$  is also not small and known with 5% accuracy, the use of this combination allows to simplify an extraction of  $\alpha_s(\mu)$  from the fine structure data.

For the fixed charm-quark mass  $m$  and NP static interaction the value of  $\alpha_s(\mu_0)$  and the scale mass  $\mu_0$  are unambiguously calculated from our fits. In practice the variation of string tension is admitted in the range  $0.17 \leq \sigma \leq 0.185 \text{ GeV}^2$  till a good fit to mass level differences is obtained. The uncertainty in  $\sigma$  value gives rise to the theoretical error  $\sim 5\%$  in  $\alpha_s(\mu)$ .

The dominant theoretical uncertainty comes from the choice of  $c$  quark pole mass which has broader range than for  $\overline{MS}$  mass,  $\bar{m}(\bar{m}^2)$ . Therefore we had to solve spinless Salpeter equation for many sets of parameters with different  $m$ .

The value of  $m = 1.48 \pm 0.08 \text{ GeV}$  was obtained in our best fit which corresponds to the  $\overline{MS}$  mass  $\bar{m} = 1.18 \pm 0.07 \text{ GeV}$  and the extracted value of the strong coupling constant is  $\alpha_s(\mu_0) = 0.38 \pm 0.03(\text{exp}) \pm 0.04(\text{theor})$  at scale  $\mu_0 = 1.0 \pm 0.2 \text{ GeV}$ . The uncertainty connected with NP effects is included in theoretical error. This value of  $\alpha_s(1 \text{ GeV})$  is rather close to that calculated in bottomonium for  $2P$  state [4], probably, because  $2P \bar{b}b$  state has the same size as  $1P$  state in charmonium.

We suggest to consider relatively small value of  $\alpha_s(1 \text{ GeV})$  as a manifestation of  $\alpha_s$  freezing and compare our calculated values of  $\alpha_s(\mu)$  at  $\mu \sim 0.7 \div 1.2 \text{ GeV}$  with theoretical formula of  $\alpha_B(\mu)$  in background perturbation theory.

The behaviour of  $\alpha_B(q^2)$  at large energy scale coincides with perturbative predictions since  $\Lambda_B^{(5)} = \Lambda^{(5)}$  that gives rise to  $\alpha_s(M_Z) = 0.119 \pm 0.002$  in two-loop approximation. For  $n_f = 3$   $\Lambda_B^{(3)} = 400_{-32}^{+34} \text{ MeV}$  is slightly larger than perturbative  $\Lambda^{(3)} = 384.4 \text{ MeV}$ . With this  $\Lambda_B^{(3)}$  the predicted value of  $\alpha_s(M_\tau) \approx 0.306_{-0.014}^{+0.015}$  at the scale of  $\tau$  lepton mass is smaller than the conventional value  $0.35 \pm 0.03$  but coincides with the prediction in renormalon chain model.

The authors are grateful to Prof. B. Bakker for possibility to use his program code and Prof. Yu.A.Simonov for fruitful discussions. This work was partially supported by Russian Foundation for Basic research (project N96-02-19184a) and the joint RFBR-DFG foundation (grant N96-02-00088g).

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FIG. 1. The running coupling constant as a function of the scale parameter  $\mu$ . The perturbative two-loop  $\alpha_s(\mu)$  with  $\Lambda^{(5)} = 237^{+26}_{-24} \text{ MeV}$ ,  $\Lambda^{(4)} = 338^{+33}_{-31} \text{ MeV}$  and  $\Lambda^{(3)} = 384.4^{+32}_{-30} \text{ MeV}$  is shown by a hatched band. The running two-loop coupling constant in background perturbation theory with the background mass  $m_B = 1.1 \text{ GeV}$ ,  $\Lambda_B^{(5)} = \Lambda^{(5)}$ ,  $\Lambda_B^{(4)} = 339.2^{+33}_{-31} \text{ MeV}$  and  $\Lambda_B^{(3)} = 399.6^{+34}_{-32} \text{ MeV}$  is shown by a dashed lines. The values of  $\alpha_s(\mu_0)$  extracted from the fit to charmonium fine structure data for different quark masses are depicted by open circles with error bars including both theoretical and experimental uncertainties.

TABLE I. The comparison of  $1P$  state matrix elements for spinless Salpeter and Schrödinger Equations.

matrix element	Set A $m = 1.20 \text{ GeV}$ $\sigma = 0.20 \text{ GeV}^2$ $\tilde{\alpha}_V = 0.350$		Set B $m = 1.40 \text{ GeV}$ $\sigma = 0.17 \text{ GeV}^2$ $\tilde{\alpha}_V = 0.373$		Set C $m = 1.48 \text{ GeV}$ $\sigma = 0.18 \text{ GeV}^2$ $\tilde{\alpha}_V = 0.420$	
	R <sup>a)</sup>	NR <sup>a)</sup>	R	NR	R	NR
$\langle r^{-3} \rangle_{1P} (\text{GeV}^3)$	0.117	0.080	0.118	0.086	0.1423	0.1014
$\langle r^{-1} \rangle_{1P} (\text{GeV})$	0.382	0.345	0.382	0.351	0.4040	0.3704
$a_{NP} (\text{MeV})$	-26.53	-23.96	-16.57	-15.22	-16.60	-15.22
$\sqrt{\langle r^2 \rangle}_{1P} (\text{GeV}^{-1})$	3.281	3.565	3.293	3.514	3.130	3.340

a) R (NR) refers correspondingly to relativistic (nonrelativistic) case.

TABLE II. The spin-orbit and tensor  $1P$  state splittings and spin-averaged mass level differences in charmonium <sup>a)</sup> (in  $\text{MeV}$ ).

	Experimental values <sup>b)</sup> ( $\text{MeV}$ )	Set C $m = 1.48 \text{ GeV}$ , $\sigma = 0.18 \text{ GeV}^2$ , $\alpha_s(\mu_0) = 0.365$ , $\mu_0 = 0.909 \text{ GeV}$ , $\tilde{\alpha}_V = 0.42$	Set D $m = 1.40 \text{ GeV}$ , $\sigma = 0.183 \text{ GeV}^2$ , $\alpha_s(\mu_0) = 0.312$ , $\mu_0 = 1.357 \text{ GeV}$ , $\tilde{\alpha}_V = 0.39$
$a_{NP}$	—	-16.60	-18.40
$a_P^{(1)}$	—	47.56	41.84
$a_P^{(2)}$	—	3.61	11.12
$a_{total}$	$34.56 \pm 0.19$	34.56	34.56
$c_P^{(1)}$	—	31.70	27.89
$c_P^{(2)}$	—	7.42	11.23
$c_{total}$	$39.12 \pm 0.62$	39.12	39.12
$M(2S) - M(1S)$	$595.39 \pm 1.91$	591.30	586.09
$M(1P) - M(1S)$	$457.92 \pm 1.0$	460.68	445.93
$M(2S) - M(1P)$	$137.47 \pm 1.77$	130.62	140.15

a) all matrix elements defining spin effects were calculated for spinless Salpeter equation.

b)  $M(nL)$  means the spin-averaged mass.



